Longitudinal Vibration of Gravity-Stabilized, Large, Damped Spacecraft Modeled as Elastic Continua

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Longitudinal vibration of gravity-stabilized, structurally damped, large flexible satellites undergoing pitching librations in circular and elliptic orbits is studied. Stability analysis of the linearized system indicates 1) that vibrational stability is ensured if $p>\sqrt{3}$, and 2) that structural damping plays a role in exciting the pitch motion in eccentric orbits. Analytical expressions are derived for stability boundary and response of vibration in even-numbered modes in circular orbits using perturbation methods. They indicate that parametric resonance can occur at two additional frequencies, and the minimum (critical) damping necessary to avoid vibrational instability is sensitive to the satellite inertia ratio. An attempt is made to establish stability of the nonlinear system using the Hamiltonian of the corresponding conservative system. The stability criterion thus obtained gives a conservative estimate of the stability boundary. Numerical simulation verifies the results of the stability analysis and indicates that parametric resonance occurs only in the even-numbered modes. It is also found that the nonlinear effects arising out of vehicle flexibility do not seem to produce significant deviation either in rigid-body motion or stable vibrations.

Nomenclature

A_1 = modal amplitude $=A_1/\ell$
$A_{1m}, A_{1\ell}, A_{1e}$ = maximum, allowable, and equilibrium values of A_1
a_N, b_N = constants of motion
c = nondimensional disturbance parameter in pitch, θ'_{10}
e = orbital eccentricity
H^1_{xx} = modal integral for one mode
I_x, I_y, I_z = moments of inertia about the principal axes x, y, and z, respectively
K $=(I_y-I_x)/I_z$
K_1 $=(3K)^{1/2}$
K_2 $=H^1_{xx}\ell/I_z$
K_3 $=M\ell^2/I_z$
ℓ = length of the satellite
M = modal mass
n = mean orbital rate
p $=\omega/n$
p_N = Nth perturbation in p
$R; R_x, R_y, R_z$ = distance of the Earth's center from the satellite mass center; its components along the x, y, and z axes, respectively
t = time
ζ, ζ_c = modal damping factor, critical damping
θ = true anomaly
$\theta_1, \theta_{10}, \theta_{1e}$ = pitch angle and its initial and equilibrium values, respectively
τ $=K_1\theta+d_2$
ω = structural frequency
$\omega_x, \omega_y, \omega_z$ = components of the body angular rate about the x, y, and z axes, respectively

Superscripts

$(\dot{\ }), (\ )'$ = differentiation with respect to t and θ (or τ), respectively

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Introduction

WITH the availability of the space transportation system, large space structures having dimensions of hundreds of meters to several kilometers are now being contemplated for potential future space missions identified in Ref. 1. These spacecraft characterized by distributed mass and elasticity properties are extremely flexible and, thus, are highly susceptible to on-board or environmental disturbances causing vibrations. The present paper studies the longitudinal vibration behavior of gravity-stabilized, structurally damped, flexible spacecraft adopting a continuum model recently proposed in the literature.[2,3]

An early study on the system dynamics modeled as elastic continua[4] shows that the axial oscillations of a rotating-cable, counterweight space system undergoing planar motion are governed by the damped Mathieu equations. Numerical simulation[5] of the system indicates that the spin rate decreases gradually, but very slowly. Paul[6] carries out a linear system analysis of an extensible dumbbell satellite incorporating a viscous damper to damp the pitch motion in circular orbit. The extensible dumbbell satellite represents the earliest example of a single-degree-of-freedom model for the longitudinal vibration of a beam-like satellite. In a later paper, Ashley[7] concludes that the influence of infinitely small deformations on the moments of inertia is negligible for the flexural vibrations, however, not for the longitudinal vibrations of the beam-like satellites.

In recent years, extensive studies on the flexural vibrations have been conducted by Kumar and Bainum[8] and Maharana and Shrivastava.[9-11] These authors deal with gravity-stabilized and torque-free systems and apply Floquet theory, phase plane methods, and perturbation methods. Although a better understanding of the flexural vibration currently is available, a numerical simulation study[12] sheds light on the highly nonlinear behavior of the system. The pitch motion is noted to be negligibly affected in an initially undeformed beam. In contrast, large-amplitude pitch motion and sustained oscillations of complex waves in the flexural vibration are noticed in an initially deformed beam.

From the foregoing it is clear that the present understanding of the longitudinal vibrations is inadequate. In this paper an attempt has been made to fill this gap. Treating the system as an elastic continuum, it is shown that the vibrations and libration can be described by two coupled, second-order, ordinary nonlinear differential equations. The response and stability of these equations are analyzed next.

For the linearized system, it is found that the motion is stable if $p>(3)^{1/2}$ and that the structural damping excites the pitch motion in eccentric orbits. More accurate analytical expressions of the stability conditions and the response of vibration in the even-numbered modes are obtained by applying the method of strained parameters and the multiple-scale approach, respectively. Parametric resonance is found to occur at two additional frequencies. The critical damping required to avoid instability is found to be a function of the satellite inertia ratio. For the highly nonlinear conservative system, criteria for the "stability-in-the-large" are offered using the system Hamiltonian. Numerical solutions of the equations of motion confirm the results obtained by the analytical methods.

Formulation

The equations of motion of a flexible spacecraft modeled as an elastic continuum describing the orbital, attitude, and structural motion have been derived by several authors[7,8,13] following the Newton-Euler approach. After introducing such realistic assumptions as 1) the negligible influence of attitude or structural motion on the orbital motion, 2) a spherical Earth, 3) negligible coupling among the modes of vibration, and 4) a viscous model of structural damping, one obtains the following set of the equations of motion:

$$
\begin{aligned}
&I_x\dot{\omega}_x+(I_z-I_y)\omega_y\omega_z=3\Omega^2[R_yR_z(I_z-I_y)/R^2]\\
&(I_y+2A_1H^1_{xx}+A_1^2M)\dot{\omega}_y+(I_x-I_z-2A_1H^1_{xx}-A_1^2M)\omega_z\omega_x\\
&\quad=3\Omega^2[R_zR_x(I_x-I_z-2A_1H^1_{xx}-A_1^2M)/R^2]\\
&\quad-2\omega_y(\dot{A}_1H^1_{xx}+A_1\dot{A}_1M)\\
&(I_z+2A_1H^1_{xx}+A_1^2M)\dot{\omega}_z+(I_y-I_x+2A_1H^1_{xx}-A_1^2M)\omega_x\omega_y\\
&\quad=3\Omega^2[R_xR_y(I_y-I_x+2A_1H^1_{xx}+A_1^2M)/R^2]\\
&\quad-2\omega_z(\dot{A}_1H^1_{xx}+A_1\dot{A}_1M)\\
&M\ddot{A}_1+2M\zeta\omega\dot{A}_1+[\omega^2-\omega_y^2-\omega_z^2+\Omega^2-3\Omega^2R_x^2/R^2]MA_1\\
&\quad=-\Omega^2H^1_{xx}+3\Omega^2R_x^2H^1_{xx}/R^2+(\omega_y^2+\omega_z^2)H^1_{xx}
\end{aligned}
\tag{1}
$$

where Ω is the orbital rate. The first three equations represent the attitude motion and the last one corresponds to the longitudinal vibration of each mode of the structure. The first- and second-order effects of vibration on the attitude motion are evident from these equations. The complexity prevents one from carrying out a general study of these equations. Therefore, attention is restricted to the pitch motion only, which is the most dominant and can get excited by itself decoupled from roll-yaw motion. This simplifies the above set of equations reducing them to two ordinary differential equations which describe the pitch motion and longitudinal vibration of the satellite. After introducing the laws of Keplerian orbit and using the nondimensional quantities (henceforth, A_1 is nondimensional) defined in the notation, these can be written as

$$
\begin{aligned}
&(1+2K_2A_1+K_3A_1^2)\theta_1''+[-2e\sin\theta(1+2K_2A_1\\
&\quad+K_3A_1^2)/(1+e\cos\theta)+2(K_2A_1'+K_3A_1A_1')]\theta_1'\\
&\quad+3(K+2K_2A_1+K_3A_1^2)\cos\theta_1\sin\theta_1/(1+e\cos\theta)\\
&\quad=2e\sin\theta(1+2K_2A_1+K_3A_1^2)/(1+e\cos\theta)\\
&\quad-2(K_2A_1'+K_3A_1A_1')
\end{aligned}
\tag{2a}
$$

$$
\begin{aligned}
&A_1''+[2\zeta p(1-e^2)^{1.5}/(1+e\cos\theta)^2-2e\sin\theta/(1+e\cos\theta)]A_1'\\
&\quad+[p^2(1-e^2)^3/(1+e\cos\theta)^4-1-\theta_1'^2-2\theta_1'\\
&\quad-(3\cos2\theta_1+1)/\{2(1+e\cos\theta)\}]A_1\\
&\quad=[(3\cos2\theta_1+1)/\{2(1+e\cos\theta)\}+1+\theta_1'^2+2\theta_1']K_2/K_3
\end{aligned}
\tag{2b}
$$

Despite the simplifications, these equations of motion are highly coupled, nonlinear, and nonautonomous in an elliptic orbit, implying that vibration and libration significantly influence each other. In circular orbits the equations are given by

$$
\begin{aligned}
&(1+2K_2A_1+K_3A_1^2)\theta_1''+2(K_2A_1'+K_3A_1A_1')\theta_1'\\
&\quad+3(K+2K_2A_1+K_3A_1^2)\cos\theta_1\sin\theta_1\\
&\quad=-2(K_2A_1'+K_3A_1A_1')
\end{aligned}
\tag{3a}
$$

$$
\begin{aligned}
&A_1''+2\zeta pA_1'+[p^2-1-\theta_1'^2-2\theta_1'-(3\cos2\theta_1+1)/2]A_1\\
&\quad=[(3\cos2\theta_1+1)/2+1+\theta_1'^2+2\theta_1']K_2/K_3
\end{aligned}
\tag{3b}
$$

As expected, Eqs. (3) represent a set of autonomous, nonlinear, ordinary differential equations and, therefore, possess equilibrium points. In the case of longitudinal vibration taking place in the even-numbered modes, it is known that K_2 vanishes, simplifying Eqs. (3) further. For convenience they are recorded below.

$$
\begin{aligned}
&(1+K_3A_1^2)\theta_1''+2K_3A_1A_1'\theta_1'+3(K+K_3A_1^2)\cos\theta_1\sin\theta_1\\
&\quad=-2K_3A_1A_1'
\end{aligned}
\tag{4a}
$$

$$
A_1''+2\zeta pA_1'+[p^2-1-\theta_1'^2-2\theta_1'-(3\cos2\theta_1+1)/2]A_1=0
\tag{4b}
$$

Therefore, unlike the vibration in the odd-numbered modes, vibration in the even-numbered modes is not excited externally by the pitch motion. Before analyzing the nonlinear equations (3), it is worthwhile to investigate the stability of linearized motion about the equilibrium points.

Variational Stability

Setting all derivatives of the motion variables to zero in Eqs. (3), the equilibrium points are found to be

$$\theta_{1e}=n\pi/2 \quad \text{with} \quad n=0,1$$

and

$$
\begin{aligned}
A_{1e}&=3K_2/\{K_3(p^2-3)\} && \text{for } n=0\\
&=0 && \text{for } n=1
\end{aligned}
\tag{5}
$$

These are the same as the equilibrium points corresponding to a rigid satellite and must be valid for a flexible satellite for which $p\neq\sqrt{3}$. At the equilibrium point, the satellite remains under tension due to the action of gravitational and centrifugal forces. Now the variational equations about these equilibrium points can be obtained readily by substituting

$$A_1=A_{1e}+x \quad \text{and} \quad \theta_1=\theta_{1e}+y \tag{6}$$

into Eqs. (2) and neglecting the nonlinear terms, as

$$y'' + 3K_4\cos(n\pi)y + 2K_5x' = 2e\sin\theta \tag{7a}$$

$$\begin{aligned} x'' &+ 2\zeta p x' + [p^2 - 3\{\cos(n\pi)+1\}/2]x - 2(A_{1e}+K_2/K_3)y' \\ &= -[p^2 - 3\{\cos(n\pi)+1\}/2]A_{1e} + 3K_2\{\cos(n\pi) \\ &+ 1\}/(2K_3) + [4p^2A_{1e} + (A_{1e}+K_2/K_3) \\ &\times \{3\cos(n\pi)+1\}/2]e\cos\theta \end{aligned} \tag{7b}$$

Note that a small deviation from circular orbit has been assumed and K_4 and K_5 are given by

$$\begin{aligned} K_4 &= [K_3K(p^2-3)^2 + 3K_2^2(2p^2-3)] \\ &\div [K_3(p^2-3)^2 + 3K_2^2(2p^2-3)] \\ K_5 &= p^2K_2K_3/[K_3(p^2-3)^2 + 3K_2^2(2p^2-3)] \quad \text{for } n=0 \end{aligned} \tag{8a}$$

and

$$K_4 = K \quad \text{and} \quad K_5 = K_2 \qquad \text{for } n=1 \tag{8b}$$

Looking at the structure of Eqs. (7) one finds that structural motion acts as a damper to the pitch motion which, in turn, produces a gyroscopic effect on vibration. Equations (7) are a set of two linear, second-order, ordinary differential equations that will be more revealing if they are cast into two fourth-order differential equations in θ_1 and A_1. Only the equation corresponding to the pitch motion is given below.

$$\begin{aligned} y^{iv} &+ 2\zeta p y^{iii} + [3K_4\cos(n\pi) + [p^2 - 3\{\cos(n\pi)+1\}/2] \\ &+ 4K_5(A_{1e}+K_2/K_3)]y'' + 6K_4\zeta p\cos(n\pi)y' \\ &+ 3K_4\cos(n\pi)[p^2 - 3\{\cos(n\pi)+1\}/2]y \\ &= 2[p^2 - \{3\cos(n\pi)+1\}/2]e\sin\theta + 4e\zeta p\cos\theta \\ &+ 2K_5[4p^2A_{1e} + (A_{1e}+K_2/K_3)\{3\cos(n\pi)+1\}/2]e\sin\theta \end{aligned} \tag{9}$$

The constant coefficients of the differential equation are the coefficients of the characteristic equation. A straightforward application of the Routh-Hurwitz criteria yields the condition for stability-in-the-small as

$$p > \sqrt{3} \quad \text{for} \quad n=0 \tag{10}$$

and, a criterion for stability does not exist for $n=1$ with the present system. The above condition then implies that the elastic vehicle is rigid-body stable if $p>\sqrt{3}$. In other words, if a vehicle's structural frequency is less than or equal to $\sqrt{3}$ times the mean orbital rate, rigid-body stability in planar libration no longer holds. System stability does not depend upon any other satellite characteristics, although K_2 and K_3 do influence the response of the system.

For resonance to occur, it can be shown that K must be equal to 1/3. Such a value of inertia ratio is not possible for the present satellite configurations with a thin structure. Therefore, resonance is ruled out for the systems under consideration.

From Eq. (9), it can be observed that vibration as well as libration are externally forced due to the ecentricity of the orbit. Equation (9) further reveals that a part of libration is caused due to the presence of structural damping. Thus, structural damping does excite the pitch motion in elliptic orbits. Although the effect is of second order (since its amplitude is proportional to $e\zeta$), it does indicate that structural dampers used to damp out the pitch libration are effective only in circular orbits. Therefore, the practical utility of such dampers is limited.

The response of a similar system has been analyzed thoroughly by Paul[6] and will not be repeated here.

Vibration in Even-Numbered Modes

It is noted that Eqs. (4) have the same structure as those representing inplane flexural vibration during pitching in a circular orbit.[11] In Ref. 11 the authors have analyzed stability and response of flexural vibration using perturbation methods. These methods are now applied to this case. Before carrying out the analysis, Eqs. (4) need to be simplified.

Equation (4a), as well as some past studies,[7-10] clearly show that the influence of vibration on the vehicle's moment of inertia is of second order and, in the light of small deformation theory, can be neglected. Assuming that this is so in the present case, one finds that Eq. (4) representing pitch motion is decoupled from Eq. (4b), representing governing vibrations. Further, restricting the analysis to small pitching oscillations, Eq. (4a) yields a solution for the rigid-body motion in the form of simple harmonic motion given by

$$\theta_1 = d_1\sin\tau \tag{11}$$

where

$$d_1 = (\theta_{10}^2 + \theta'^2_{10}/3K)^{1/2}, \qquad d_2 = \tan^{-1}(\theta_{10}K_1/\theta'_{10})$$

With this approximation Eq. (4b) is also modified to give

$$A_1'' + 2\zeta p A_1' + (p^2 - 3 - \theta'^2_1 - 2\theta_1' + 3\theta_1^2)A_1 = 0 \tag{12}$$

Substitution of Eq. (12) into Eq. (13) yields, after a change of variable from θ to τ, the following relation for impulsive pitch motion:

$$\begin{aligned} \frac{d^2A_1}{d\tau^2} &+ (2\zeta p/K_1)\frac{dA_1}{d\tau} + \{2K(p^2-3) + (1-K)c^2 \\ &- 4Kc\cos\tau - (1+K)c^2\cos 2\tau\}A_1/(6K^2) = 0 \end{aligned} \tag{13}$$

Equation (13) is recognized as Hill's three-term equation or Whittakar's equation, the stability of which can be carried out by the application of Floquet theory. In this approach, however, one has to integrate the equation numerically. Instead, closed-form solutions for Eq. (14) will be developed herein using perturbation methods.

Stability Analysis

A common practice is to study stability under small disturbances. The method of strained parameters[16]—one of several perturbation techniques—is considered to be an efficient approach. The method requires expansion for the solution and the system frequency in a power series of the perturbation parameter. The coefficients are determined by eliminating secular terms. In practice, it is sufficient to truncate the series to the second-order term. Accordingly, the following expansions are assumed for the subsequent analysis:

$$A_1 = A_{10} + cA_{11} + c^2A_{12} \tag{14}$$

$$p^2/3K + (1-K)c^2/(6K^2) = p_0^2 + cp_1 + c^2p_2 \quad \text{or}$$

$$p/K_1 = p_0 + p_1c/(2p_0) + [p_2 + (1-K)/6K^2]c^2/(2p_0) \tag{15}$$

Granted that these expansions are valid, one is interested in the effects of a small amount of structural damping, i.e., damping of the first order $\zeta = \zeta_1 c$.

Substituting Eqs. (14) and (15) into Eq. (13) and equating the coefficients of equal powers of c to zero, one obtains a set of linear, constant-coefficient, second-order differential equations. The equation corresponding to the c^0 coefficient is solved to yield

$$A_{10} = a_0 \cos(p_{0m}\tau) + b_0 \sin(p_{0m}\tau) \tag{16}$$

where $p_{0m}^2 = (p_0^2 - 1/K)$, and a_0 and b_0 are constants depending upon the initial conditions of vibration. This is the zeroth-order solution. Using this in the equation corresponding to c^1, one can determine A_{11}. For a uniformly valid expansion to exist in the primary zone of resonance, $p_{0m} = ½$, the terms giving rise to secular response in A_{11} must vanish. This leads to the following expresssions valid on the transition curves:

$$p_1 = \pm [1/9K^2 - \zeta_1^2 (1/4 + 1/K)]^{1/2}$$

and

$$a_0 = (p_1 + 1/3K) b_0 / (p_0 \zeta_1) \tag{17}$$

With these, the equation for A_{11} is solved to give

$$A_{11} = -[a_0 \cos(p_{0m}+1)\tau\} + b_0 \sin\{(p_{0m}+1)\tau\}]$$

$$\div \{3K(1 + 2p_{0m})\} - a_1 \cos(p_{0m}\tau) + b_1 \sin(p_{0m}\tau) \tag{18}$$

Following the same procedure one determines A_{12} from the equation obtained by comparing coefficients of c^2 and using A_{10} and A_{11}. Invoking the condition of periodicity of the solution A_{12} at $p_{0m} = ½$, yields

$$p_2 = p_1^2/(4p_0^2) - \zeta_1^2/4 - (3K+4)/(36K^3) \tag{19}$$

Combining Eqs. (15), (17), and (19), one obtains the following relation for the transition curves near the primary zone of resonance ($p_{0m} = ½$), to the second order:

$$p = K_1 [(1/4 + 1/K)^{½} \pm \{c^2/9K^2 - \zeta^2(1/4 + 1/K)\}^{½} / \{2(1/4$$

$$+ 1/K)^{½}\} - \{\zeta^2(1+K)/4K + (6K^3 + 21K^2 - 12K + 16)c^2$$

$$\div \{36K^3(4+K)\}/\{2(1/4 + 1/K)^{½}\}] \tag{20}$$

The right-hand side of the above equation would have real values only if the expression under the square root were a positive quantity. This implies that the maximum disturbance, c_{max} to be tolerated by the structure with stable vibration near the primary resonance is

$$c_{max} = 3K\zeta(1/4 + 1/K)^{½} \tag{21}$$

A more important quantity is the critical damping necessary to avoid vibrational instability for a given disturbance level. This can be derived as

$$\zeta_c = c/[3K(1/4 + 1/K)^{½}] \tag{22}$$

The above equation reveals a complicated dependence of critical damping on the vehicle inertia ratio.

For $p_{0m} = ½$, it can be shown that the determinant of the system formed for a_0 and b_0 can never be equal to zero for any real values of ζ or p_{0m} in order to eliminate secular terms. This implies that there does not exist a stability boundary. Thus, first-order damping removes instability regions near all other resonance conditions that may be present in an undamped system. The incorporation of damping higher than first order will at best give transition curves defined by Eq. (20). Therefore, it remains to investigate the effects of the second-order damping given by $\zeta = \zeta_1 c^2$. Following the same steps as before, one obtains, for $p_{0m} = 1$,

$$p_1 = 0$$

and

$$p_2 = 1/(9K^2) \pm [289/(11664K^4) + (9K+34)/(1296K^3)$$

$$-4\zeta_1^2(1+K)/K]^{½} \tag{23}$$

Hence, the stability boundaries near $p_{0m} = 1$ are given by

$$p = K_1 [(1 + 1/K)^{½} \pm \{(81K^2 + 306K + 289)c^2/(11664K^4)$$

$$+ 4\zeta^2(1+K)/K\}^{½} c/\{2(1 + 1/K)^{½}\}$$

$$+ (5 - 3K)c^2/\{36K^2(1 + 1/K)^{½}\}] \tag{24}$$

and the maximum tolerable disturbance level and critical damping at this resonance are given by

$$c_{max} = \{K(1+K)\}^{½} 216K\zeta/[(81K^2 + 306K + 289)^{½}] \tag{25}$$

$$\zeta_c = (81K^2 + 306K + 289)^{½} c/[(216K(K+K^2)^{½}] \tag{26}$$

As might be expected, the critical damping requirement for the second resonance is less than that for the primary resonance. Proceeding further, it is easily shown that the second-order damping removes all resonances for $p_{0m} \geq 2$ in the case of small disturbances.

Attention is now drawn to find instability near the zone of $p_{0m} = 0$. Note that $p_0 = 1/\sqrt{K}$ in this case. On carrying out the same steps as for the analysis with the first-order damping, the stability boundary is now found to be

$$p = \sqrt{3}[1 - (1 + 3K)c^2/(36K)] \tag{27}$$

This equation shows that stability near $p_{0m} = 0$ or $p_0 = 1/\sqrt{K}$ is not affected by the presence of structural damping. The same conclusion can be reached even when the analysis includes full damping. Noting that for a nonsecular solution p_{0m} cannot be negative; one finds that the above frequencies are the only frequencies near instability. Discussion on the above transition curves and instability zones is presented in a later section.

Response Analysis

The study of the dynamic response of a flexible spacecraft is important. The technique used to find the stability domain gives solutions only on the transition curves. To find the response of a damped system, such as the one governed by Eq. (14), within the stable region away from the boundary, the multiple-scales technique is chosen. For this purpose, let

$$\delta^2 = (p^2 - 3)/3K + (1 - K)c^2/(6K^2) \quad \text{and} \quad \zeta = \zeta_1 c$$

and let the expansion representing the response be the following function of different time scales, T_N:

$$A_1(\tau, c) = A_{10}(T_0, T_1, T_2) + cA_{11}(T_0, T_1, T_2)$$

$$+ c^2 A_{12}(T_0, T_1, T_2) \tag{28}$$

where $T_n = c^N \tau$, $N = 0, 1, 2, \ldots$, are the new independent variables. Since the expansion up to the second order in c is considered to give a sufficiently accurate response, the derivatives with respect to τ can be written as

$$\frac{d}{d\tau} = D_0 + cD_1 + c^2 D_2; \quad D_N = \frac{\partial}{\partial T_N}$$

$$\frac{d^2}{d\tau^2} = D_0^2 + 2cD_0D_1 + c^2(D_1^2 + 2D_0D_2) \tag{29}$$

Substituting Eqs. (28) and (29) into Eq. (13), and equating the coefficients of equal powers of c to zero, a set of linear constant-coefficient, second-order differential equations is obtained. For δ not equal to zero, the solution of the equation corresponding to the coefficient of c can be written conveniently as

$$A_{10} = a_0(T_1,T_2)e^{i\delta T_0} + \bar{a}_0(T_1,T_2)e^{-i\delta T_0} \tag{30}$$

where $i=(-1)^{1/2}$; and a_0 and $\bar{a}_0$, complex conjugate to each other, are to be determined as functions of T_1 and T_2 constraining A_{11} and A_{12} to be periodic in T_0. The right-hand side of the equation for A_{11}, obtained by comparing the coefficients of c, contains secular terms. Using Eq. (30), it is found that, for δ away from 1/2, the secular terms could be eliminated if

$$a_0 = a_{01}(T_2)\exp(-\zeta_1 pT_1/K_1) \tag{31}$$

With this, the particular solution for A_{11} is given as

$$\begin{aligned} A_{11} = \exp(-\zeta_1 pT_1/K_1)[&-a_{01}\exp\{i(\delta+1)T_0\} \\ &\div(2\delta+1) + a_{01}\exp\{i(\delta-1)T_0\} \\ &\div(2\delta-1) + \bar{a}_{01}\exp\{-i(\delta-1)T_0\}/(2\delta-1) \\ &-\bar{a}_{01}\exp\{-i(\delta+1)T_0\}/(2\delta+1)]/3K \end{aligned} \tag{32}$$

Substituting Eq. (32) into the equation corresponding to the coefficients of c^2 and eliminating the secular terms for the solution, for δ away from unity, one gets

$$a_{01} = a\exp\{-i[[2/\{3K(4\delta^2-1)\} + \zeta_1^2p^2]T_2/(6K\delta) + b]\} \tag{33}$$

where a and b are now real constants to be determined from the initial conditions. The particular solution of A_{12} is then found to be

$$\begin{aligned} A_{12} = \exp(-\zeta_1 pT_1/K_1)[&E_1a_{01}\exp\{i(\delta+2)T_0\} \\ &+E_2a_{01}\exp\{i(\delta-2)T_0\} \\ &+E_2\bar{a}_{01}\exp\{-i(\delta-2)T_0\} + E_1\bar{a}_{01}\exp\{-i(\delta+2)T_0\}] \end{aligned} \tag{34}$$

where

$$E_1 = [1/\{9(2\delta+1)\} - (1+K)/12]/\{K^2(4\delta+4)\}$$

$$E_2 = [1/\{9(2\delta-1)\} + (1+K)/12]/\{K^2(4\delta-4)\}$$

After combining Eqs. (34), (32), (30), and (28), and expressing the complex exponentials in the form of trigonometric functions, the stable solution of Eq. (14), for δ away from 0, 1/2, and 1, is given by

$$\begin{aligned} A_1 = a\exp(-\zeta p\tau/K_1)[&\cos(\omega_m\tau-b) - c\cos\{(\omega_m+1)\tau-b\} \\ &\div\{3K(2\delta+1)\} + c\cos\{(\omega_m-1)\tau-b\}/\{3K(2\delta-1)\} \\ &+[1/\{3(2\delta+1)\} - (1+K)/4]c^2\cos\{(\omega_m+2)\tau-b\} \\ &\div\{12K^2(\delta+1)\} + [1/\{3(2\delta-1)\} \\ &+(1+K)/4]c^2\{(\omega_m-2)\tau-b\}/\{12K^2(\delta-1)\}] \end{aligned} \tag{35}$$

where

$$\omega_m = \delta - [2c^2/\{3K(4\delta^2-1)\} + \zeta^2p^2]/(6K\delta)$$

which can be termed as nondimensional mean frequency to the order of c^2. The response contains subharmonics and superharmonics having dominant components of the mean frequency. It is also observed that the solution exhibits an exponentially decaying behavior. This is well known from the theory of linear damped systems.

This completes the analysis of vibration of even-numbered modes in circular orbits. It may be noted that a similar analysis could also be performed in elliptical orbits with small eccentricity. However, the preceding expressions take a much more complex form. Reference 11 deals with such an analysis of the in-plane flexural vibration in elliptical orbits that can be modifed suitably to include the present case.

Stability of the Nonlinear System

The motion of a large, damped, flexible spacecraft moving in elliptic orbit and undergoing pitch libration and longitudinal vibration in the orbital plane represents a highly nonlinear, nonconservative dynamical system. To gain some insight into this problem, it seems logical to deal with the corresponding conservative system. Therefore, the nonlinear analysis is restricted to undamped systems in circular orbits. For a single mode of vibration, the gravitational potential and kinetic energies, after removing the orbital part which is assumed to remain unperturbed, are given by[14]:

$$\begin{aligned} V_G = -3n^2(I_y-I_x)\cos^2\theta_1/2 + n^2(1-3\cos^2\theta_1)(A_1^2M \\ +2A_1H_{xx}^1)/2 \end{aligned} \tag{36}$$

$$T = (I_z + A_1^2M + 2A_1H_{xx}^1)(\dot{\theta}+\dot{\theta}_1)^2/2 + \dot{A}_1^2M/2 \tag{37}$$

and the elastic potential energy for each mode is

$$V_{EL} = M\omega^2A_1^2/2 \tag{38}$$

This being an unnatural system, its Hamiltonian can be written as[14]

$$H = T_2 + V - T_0 \tag{39}$$

where

$$T_2 = (I_z + A_1^2M + 2A_1H_{xx}^1)\dot{\theta}_1^2/2 + \dot{A}_1^2M/2$$

$$T_0 = (I_z + A_1^2M + 2A_1H_{xx}^1)n^2/2; \qquad V = V_G + V_{EL} \tag{40}$$

For conservative systems, the Hamiltonian is a constant of the motion. Hence, from Eqs. (39) and (40),

$$\begin{aligned} (I_z + A_1^2M + 2A_1H_{xx}^1)\dot{\theta}_1^2 + \dot{A}_1^2M - (I_z + A_1^2M + 2A_1H_{xx}^1)n^2 \\ +M\omega^2A_1^2 - 3n^2(I_y-I_x)\cos^2\theta_1 \\ +n^2(1-3\cos^2\theta_1)(A_1^2M + 2A_1H_{xx}^1) = 2H \end{aligned} \tag{41}$$

Using the previously indicated nondimensional quantities, Eq. (41) can be rewritten as (henceforth A_1 is nondimensional)

$$\begin{aligned} (1 + K_3A_1^2 + 2K_2A_1)\theta_1'^2 + K_3A_1'^2 - 1 - 3K\cos^2\theta_1 \\ -3\cos^2\theta_1(K_3A_1^2 + 2K_2A_1) + K_3p^2A_1^2 = 2H(I_zn^2) \end{aligned} \tag{42}$$

Therefore, H can be computed from the initial conditions. For motion due to an impulsive disturbance, i.e., $A_{10} = \theta_{10} = 0$,

$$2H/(I_zn^2) = \theta_{10}'^2 + K_3A_{10}'^2 - 1 - 3K \tag{43}$$

Since T_2 is always positive definite, for motion to exist the motion will be bounded by [from Eq. (39)]

$$(V - T_0) \leq H \tag{44}$$

The equality sign in Eq. (44) gives the equation of the zero-velocity curves. After using Eqs. (37), (38), (40), and (43), and nondimensionalizing, Eq. (44) can be written as

$$[K_3p^2A_1^2 - 3K\cos^2\theta_1 - 3\cos^2\theta_1\,(K_3A_1^2 + 2K_2A_1)]$$
$$\le (\theta'^2_{10} + K_3A'^2_{10} - 3K) \qquad (45)$$

The expression on the left-hand side of Eq. (45) is a function of the modal structural deflection A_1 and the pitch angle θ_1 at each point of time as the motion proceeds. Taking the equality sign in Eq. (45), one can obtain A_1 as

$$A_1 = [-6K_2\cos^2\theta_1 \pm [36K_2^2\cos^4\theta_1 + 4[3K(1-\cos^2\theta_1)$$
$$-\theta'^2_{10} - K_3A'^2_{10}] \times [3K_3\cos^2\theta_1 - p^2K_3]]^{1/2}]$$
$$\div [2K_3(3\cos^2\theta_1 - p^2)] \qquad (46)$$

A simple differentiation shows that A_1 is maximum at $\theta_1 = 0$. It is given by

$$A_{1m} = [3K_2 \pm [9K_2^2 + (\theta'^2_{10} + K_3A'^2_{10})(p^2-3)K_3]^{1/2}]$$
$$\div \{K_3(p^2-3)\} \qquad (47)$$

The mission requirements and structural safety dictate that the maximum value of the modal deflection be restricted to an allowable safe value, $A_{1\ell}$, i.e.,

$$A_{1m} \le A_{1\ell} \qquad (48)$$

Substituting Eq. (47) into Eq. (48) and rearranging, one gets

$$p \ge [(6K_2A_{1\ell} + \theta'^2_{10} + K_3A'^2_{10})/(K_3A_{1\ell}^2) + 3]^{1/2} \qquad (49)$$

Equation (49) readily gives a criterion for the stability (acceptable level) of vibration for given values of θ'_{10} and A'_{10}. Also, Eq. (45) leads to the following condition for the nontumbling pitching motion ($\theta_1 \le \pi/2$):

$$(\theta'^2_{10} + K_3A'^2_{10}) \le 3K \qquad (50)$$

Thus, the satisfaction of inequalities (49) and (50) assures the stability of the nonlinear system. It is appropriate to make some important observations regarding the stability criteria given above. They represent conditions for the stability-in-the-large of the present system. Inequality (49) contains the linearized system stability criterion as a special case, corroborating that the nonlinear system yields a stringent condition. This is often observed in the stability analysis of parametrically excited systems. Moreover, the absence of the vehicle inertia ratio in Eq. (49) and the structural frequency in Eq. (50) demonstrates that they have little influence on the choice of the stability boundaries, at least in a conservative sense. It can also be seen that both inequalities are predominantly influenced by the initial conditions, mainly due to pitching disturbances. Further, the rigid-body pitch stability criterion can be obtained from Eq. (50) as a special case.

Finally some comments on the Liapunov stability are in order. Strictly speaking, inequalities (49) and (50) represent boundedness of motion and not stability of the system in the sense of Lyapunov. Boundedness and stability in the sense of Lyapunov have been proven to be equivalent only for linear systems.[15] In any case, asymptotic stability is prohibited in the absence of damping. As the present system is close to a linear time-varying system, it is reasonable to anticipate, however, that the presence of structural damping would lead to asymptotic stability. This is ascertained in the next section.

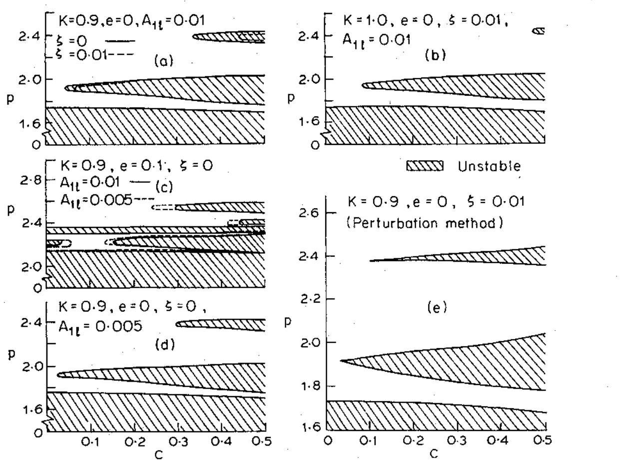

Fig. 1 Stability of longitudinal vibration in even-numbered modes; $K_3 = 6.0$ and $A'_{10} = 0.001$.

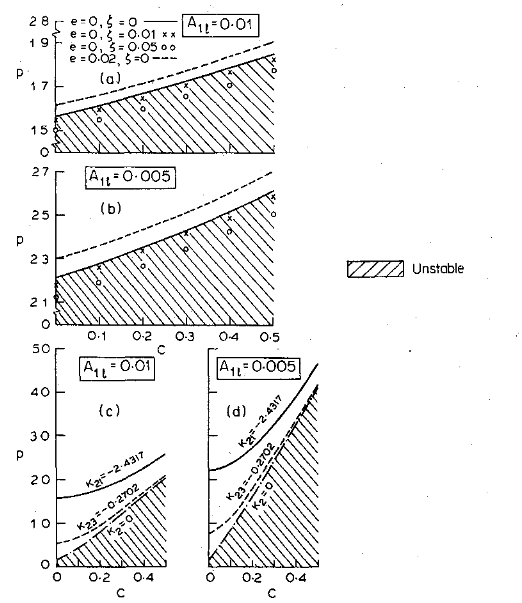

Fig. 2 Stability of longitudinal vibration in odd-numbered modes; $K = 0.9$, $K_3 = 6.0$, and $A'_{10} = 0.001$.

Parametric Study

Stability of Vibration

In the previous sections the stability and response of vibration are analyzed in an approximate fashion. To establish the validity of the results, a parametric study by numerically integrating the original nonlinear system of equations is carried out next.

The equations are integrated over 15 orbits with the help of the Runge-Kutta fourth-order algorithm. The system is said to be unstable if the modal deflection exceeds a certain prescribed value $A_{1\ell}$. This stability criterion is adopted following the analysis in the previous section. Figure 1 presents stability plots for the longitudinal vibration in even-numbered modes, in which a resolution of $\Delta p = \Delta c = 0.05$ is employed. The stability diagrams consist of a primary zone of instability nearly the same as that of the linearized system [Eq. 10] and several secondary zones of instability at higher frequencies. Due to the employment of a finite resolution, Figs. 1a, 1c, and 1d indicate the existence of a finite critical disturbance c_c, below which the instability does not occur in an undamped system. The introduction of damping (Fig. 1a) increases the critical disturbance. Structural damping does not alter the instability zones in any other way.

An increase in the inertia ratio results in an upward shift of the secondary instability zones and an increase of c_c at the higher frequency instability zones. For example, for $K=0.9$, c_c is equal to 0.44 to avoid the second zone of instability and equal to 0.065 to avoid the first instability zone; while for $K=1$, these values change to 0.475 and 0.085, respectively. Therefore, an increase in the inertia ratio aids stability near the secondary zones. The primary zone of instability remains relatively unaffected.

In the presence of orbital eccentricity, a significant upward shift of the boundary of the primary instability zone and several additional secondary zones of instability are observed (Fig. 1c). An unstable stripe of constant width and splitting of the secondary instability zone[11] are noted. Thus orbital ellipticity impairs stability in a peculiar way.

It is important to examine the effects of variation in specifying the allowable modal deflection. There is little difference in the upper and lower boundaries of the instability zones in Figs. 1c and 1d plotted for different values of $A_{1\ell}$. The value of c_c is reduced, however. Thus the stability criterion adopted serves the purpose reasonably well.

The usefulness of the stability criterion as well as the analytical solutions can be further established by comparing Figs. 1a and 1e. There is a near-perfect matching of the stability boundaries between these figures. However, critical damping is overestimated in Fig. 1a due to the employment of a finite solution. It is to be noted that the system is stable in the sense of Lyapunov in Fig. 1e, and is characterized by the bounded motion in Fig. 1a. The two figures coincide because, as shown earlier, the original system reduces to a linear system under the stipulated conditions. In other words, the system is weakly nonlinear. Therefore, it is established that the perturbation approach can describe the stability boundaries of the longitudinal vibration in the even-numbered modes quite accurately.

The stability plots for the longitudinal vibration in the odd-numbered modes obtained from the numerical solution which employs a resolution of $\Delta p=0.5$ are presented in Figs. 2a and 2b. Only the first mode is considered. Since the boundary of the instability zone occurs at a much higher frequency, the odd-numbered modes are more unstable than the even-numbered modes. Thus, the linearized system stability criterion [Eq. (10)] is no longer valid. This implies that the influence of the nonlinearity is rather strong. Further, the secondary zones of instability are absent within the scanned region of the p-c plane. However, it is not possible to establish their nonoccurrence beyond the scanned region without significant computational effort. The effects of various parameters on the system behavior follow the same trend as in the case of the even modes.

Stability boundaries resulting from the Hamiltonian analysis are plotted in Figs. 2c and 2d using Eq. (49). The curves marked with a zero value of K_I represent boundaries of longitudinal vibration in the even-numbered modes and those with nonzero values represent boundaries of longitudinal vibration in the first (corresponding to K_{21}) and third modes (corresponding to K_{23}). The range of c chosen satisfies condition (50). These curves provide a very conservative estimate of the stability boundaries, especially for large pitch disturbances. These plots also indicate that the secondary zones of instability are absent and the linearized system stability criterion (10) is not valid. It is also apparent that the criterion [(Eq. (49)] can account for small eccentricity in orbit.

The above plots are enough to expose the stability behavior of the longitudinal vibration in the even- as well as odd-numbered modes.

Response of Vibration

In this subsection a few numerical solutions of the exact nonlinear equations of motion which cannot be solved in a closed form are examined. Figure 3 presents the response of pitch, and vibration in the even-numbered modes. In circular orbits (Figs. 3a and 3b), the pitch motion remains unaffected and the structural motion decays in the presence of damping. A small deviation from the circular orbit (Fig. 3c) does not introduce any noticeable change in the stable structural motion. In this case, too, the pitch motion is unaffected by the vibration.

Figure 3d shows the undamped and damped responses for the even-numbered modes in circular orbit, obtained by applying the method of multiple scales. A comparison with their counterparts (Figs. 3a and 3b) obtained numerically indicates that the analytical solution gives responses of higher amplitude and lower frequency. The analytical approach, therefore, leads to a conservative estimate of the system response. However, it does serve the purpose of giving an insight into the system.

Figure 4 shows the pitch response of the system undergoing longitudinal vibration in the odd-numbered modes; only the first mode is considered for a spacecraft having a structural frequency of $p=20$. The effects of small amounts of structural

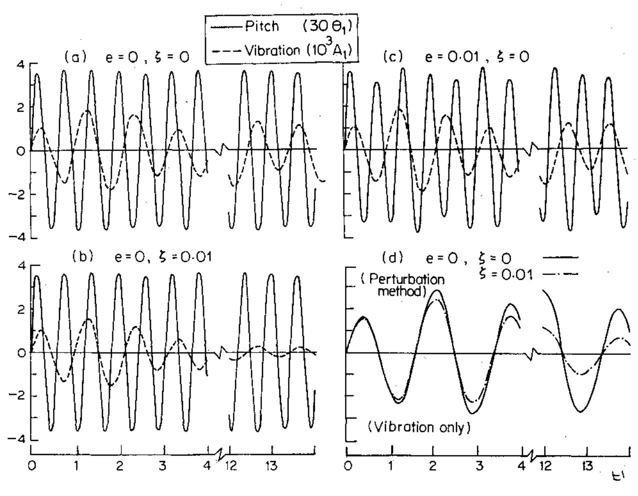

Fig. 3 Response of longitudinal vibration in even-numbered modes; $K=0.9$, $K_3=6.0$, $p=2.0$, $c=0.2$, and $A'_{10}=0.001$.

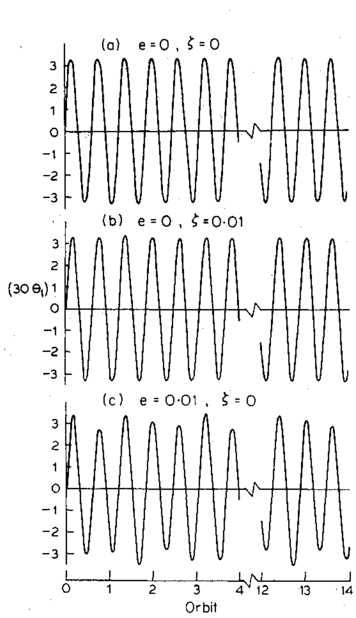

Fig. 4 Pitch response of longitudinal vibration in odd-numbered modes; $K=0.9$, $K_2=-2.4317$, $K_3=6.0$, $p=20.0$, $c=0.2$, and $A'_{10}=0.001$.

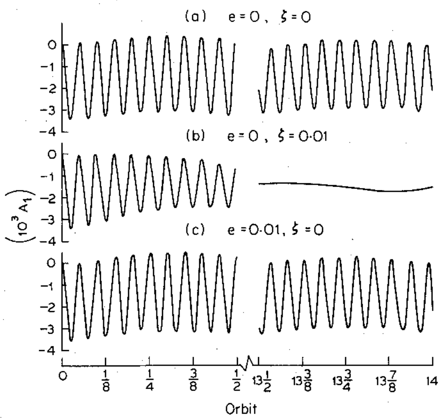

Fig. 5 Vibration response of longitudinal vibration in odd-numbered modes; $K=0.9$, $K_2=-2.4317$, $K_3=6.0$, $p=20.0$, $c=0.2$, and $A'_{10}=0.001$.

damping and orbital ellipticity are investigated. The plots show that the pitch motion essentially follows the rigid-body behavior. Therefore, the structural motion hardly affects the pitch motion in these cases.

Plots of the structural response of the above system (Fig. 5) show steady-state vibrations having negative displacements. This results from the external forcing which is compressive in nature and felt only in the vibration in the odd-numbered modes. Structural damping (Fig. 5b) helps to decay the vibration in a manner similar to that for the constant-coefficient, single-degree-of-freedom system. Due to the forcing, however, the structural motion settles down gradually to a nonzero equilibrium state.

The study also indicates that the orbital ellipticity (Fig. 5c) generally tends to increase the amplitude of vibration. This is expected because of the forced pitch motion in eccentric orbits.

Concluding Remarks

The present study is concerned with the longitudinal vibration of gravity-stabilized, large flexible satellites. Results of this study are also applicable to flexural vibration of the system in a number of configurations. The analysis of "stability-in-the-small" indicates that the condition $p>\sqrt{3}$ must be met to obtain a stable system. More accurate analysis using perturbation methods reveals the presence of parametric resonances in vibration in the even-numbered modes at two other higher frequencies and shows that critical damping is strongly dependent on the vehicle inertia ratio. A new stability criterion, valid for the stability-in-the-large, based on the boundedness of the motion, is developed using the system Hamiltonian. This approximate criterion provides a conservative estimate of the stability boundary that can be used for orbits of small eccentricity also. Numerical results validate the accuracy of these criteria. The response studies also indicate that the effect of stable longitudinal vibration on the rigid-body pitch motion is negligible. This justifies the application of small deformation theory of vibration in the analysis. The present work complements many recent investigations on the dynamics of large flexible satellites. The analytical results and parametric study should be useful to designers.

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